Inequality 3

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Q> $a, b \in \mathbb{R}$ and $0 < a \leq b \leq 1$. Prove that $0 \leq \frac{b-a}{1-ab} \leq 1$

Q> $a, b \in \mathbb{R}$ and $0 < a \leq b \leq 1$. Prove that

$$0 \leq \left(\frac{a}{1+b} + \frac{b}{1+a}\right) \leq 1$$

Q> $a, b \in \mathbb{R}$ and $0 < a \leq b \leq 1$. Prove that $0 \leq ab^2 - ba^2 \leq \frac{1}{4}$

$0 < b \leq 1 \qquad 1+a > 1$

$0 \leq b(1+a) \leq 1+a$

$\Rightarrow \quad 0 \leq b + ab \leq 1+a$

$b + ab \leq 1 + a$

$b - a + ab \leq 1$

$b - a \leq 1 - ab$

$b - a \geq 0$

$\Rightarrow \quad 0 \leq b-a \leq 1-ab$

Case 1: $1 - ab > 0$

$$\boxed{\Rightarrow \quad 0 \leq \frac{b-a}{1-ab} \leq 1}$$

So only this case will be taken

Case 2: $1 - ab = 0 \Rightarrow ab = 1 \Rightarrow a = 1, b = 1$

$\Rightarrow$ undefined

Q> If $a \geq b$ and $x \geq y$ then prove that $ax + by \geq ay + bx$

Ans:- $a - b \geq 0 \qquad x - y \geq 0$

$(a-b)(x-y) \geq 0$

$\Rightarrow \quad ax - ay - bx + by \geq 0$

$\Rightarrow \quad ax + by \geq ay + bx$

Q> If $x, y > 0$ then prove that $\left(\sqrt{\frac{x^2}{y}} + \sqrt{\frac{y^2}{x}}\right) \geq \left(\sqrt{x} + \sqrt{y}\right)$

Ans:- $\sqrt{x}, \sqrt{y} > 0 \qquad x^2 > 0$

$\sqrt{x} + \sqrt{y} > 0 \qquad \frac{x^2}{y} > 0$

$\sqrt{\frac{x^2}{y}} > 0$ and $\sqrt{\frac{y^2}{x}} > 0$

$\Rightarrow \sqrt{\frac{x^2}{y}} + \sqrt{\frac{y^2}{x}} > 0$

Hint:- WLOG, $x \geq y \Rightarrow \sqrt{x^2} \geq \sqrt{y^2}$

$$\Rightarrow \frac{1}{\sqrt{y}} \geq \frac{1}{\sqrt{x}}$$

$$\sqrt{x^2}\sqrt{\frac{1}{y}} + \sqrt{y^2}\sqrt{\frac{1}{x}} \geq \sqrt{x^2}\sqrt{\frac{1}{x}} + \sqrt{y^2}\sqrt{\frac{1}{y}}$$ (from previous solution)

$$\Rightarrow \sqrt{\frac{x^2}{y}} + \sqrt{\frac{y^2}{x}} \geq \sqrt{x} + \sqrt{y}$$

$$\sqrt{\frac{x^2}{y}} - \sqrt{x} + \sqrt{\frac{y^2}{x}} - \sqrt{y} = \underbrace{\left(\sqrt{x^2} - \sqrt{y^2}\right)}_{\geq 0}\underbrace{\left(\frac{1}{\sqrt{y}} - \frac{1}{\sqrt{x}}\right)}_{\geq 0} \geq 0$$

$$\frac{1}{x} \leq \frac{1}{y} \Rightarrow \frac{1}{\sqrt{x}} \leq \frac{1}{\sqrt{y}}$$

$$\sqrt{\frac{x^2}{y}} + \sqrt{\frac{y^2}{x}} \geq \sqrt{x} + \sqrt{y}$$

HomeWork

Q> Let a, b, c, d be real numbers with $a+d = b+c$

Prove that,

$$(a-b)(c-d) + (a-c)(b-d) + (d-a)(b-c) \geq 0$$